The current status of portfolio insurance: the history of greed and fair game

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This paper reviews the recent development of portfolio insurance. An option replication strategy is one of the optimal investment policies for the long-term investor and is widely discussed among academics and practitioners. Investor utility maximization as well as non-preference approaches have played major roles in this development. Cox's and Leland's concepts of self-financing and path-independence, and the Dybvig payoff-pricing distribution model are attractive tools to analyze the efficiency of dynamic investment strategies. Unfortunately, the financial industry does not actively use these approaches. This paper explains the basic properties of a complete market, and the efficiency and effectiveness of option replication strategies for novice researchers and practitioners. These concepts have strong relationship with the delta, gamma, and theta of options, and are extremely important for the practical application of portfolio insurance.

1 Developments in portfolio insurance

Portfolio insurance is a dynamic investment strategy that limits downside risk to the value of a reference portfolio while providing potential gains from upward trends for the price of the insurance premium. An insured portfolio holds a combination of a reference portfolio and its put option, and can be replicated by dynamically holding the reference portfolio and a risk-free asset. This is known as option based portfolio insurance (OBPI), introduced by Leland and Rubinstein [33][34], and Brennan and Schwartz [9].

The analysis of a dynamic investment strategy has assumed frictionless markets and unconstrained borrowing. Merton [36][37] solved this problem in a continuous time model under the assumption of the hyperbolic absolute risk aversion (HARA) utility. The price movement of a risky asset is assumed to follow a geometric Brownian motion with a constant mean and variance. Black

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[4] extends Merton's analysis of certain dynamic properties of the simple investment and consumption problem by assuming a completely general utility function. This led to a simple solution of protecting the value of a reference portfolio corresponding to the HARA utility function. Another classic problem is a portfolio turnpike theory that discusses the necessary and sufficient conditions for optimizing a long-term investor's utility function. The analysis was contributed by Cox and Huang [14], Hakansson [27], Huberman and Ross [28], Leland [30], and Mossin [42]. In 1986, Perold [39] extended this work and introduced constant proportion portfolio insurance (CPPI) for a fixed-income bond. Black and Jones [5] applied CPPI to the S&P 500 Index. Black and Perold [6] developed the theory of the CPPI.

In 1973, Black and Scholes [7] provided a pricing formula for options on a risky asset assuming a complete market. The Black-Scholes (BS) model is particularly useful because it is a complete general equilibrium formulation of the problem. This opened the door to asset valuation by computing the expected value of the terminal payoff based on the risk-adjusted probability[1]. It introduced the hedging strategy for holding option positions by dynamically allocating a risk-free asset and an underlying risky asset. The value of these hedged portfolios is independent of the price movements of the underlying assets. Thus the risk-free hedged portfolio earns the risk-free interest rate. This risk-neutral dynamic hedging concept replicates a put option , as demonstrated by Leland and Rubinstein [33][34], and Brennan and Schwartz [9].

As risk neutrality is used for finding the equilibrium solution for the value of options, the terminal payoff of a dynamic investment strategy can be designed independent of optimizing the investor's utility function. The preference-free approach has opened the door for a wide variety of terminal payoffs of dynamic investment strategies.

Section 2 explains the basics of portfolio insurance and classifies its dynamic investment strategies. In section 3, the complete market, less complete market, risk-neutral measure, and pricing kernel are introduced. Section 4 contains the basic properties of investment strategies for portfolio insurance. Cox and Leland's [15] idea[2] of the necessary and sufficient conditions for an efficient dynamic investment strategy such as self-financing and path-independence[3] are explained. Dybvig's[18][19] extension of Cox and Leland's idea is discussed. In section 5, the relationship between demand for portfolio insurance and investors' behavior is analyzed. Section 6 concludes the discussion.

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[1]see [12],[26],and [13].

[2]Editor's note: Although presented in the privately circulated Proceedings of the Seminar on the Analysis of Security Prices, Center for Research in Security Prices, University of Chicago, this paper has remained unpublished for a variety of reasons.

[3]see also [4]. Cox and Leland developed Black's analysis.

2 Basic strategies for downside protection

A large number of investment strategies have been available to the long-term investor seeking downside protection. The simplest one is a combination of holding a well-diversified risky portfolio and a risk-free asset; it is known as the monetary separation theorem in which

$$\text{Minimize } \sigma^2 \text{ s.t. } \mu = \omega r + (1-\omega) r_f,$$

where σ^2 and μ are, respectively, the variance and expected return of the portfolio and ω is the allocation to the risky portfolio; r and r_f denote the expected return on the risky portfolio and risk-free asset, respectively. According to the Capital Asset Pricing Model (CAPM) [44][41][35], the well-diversified risky portfolio should be a market portfolio.

Since a static investment strategy only provides limited downside protection and upside potential, Leland and Rubinstein [33] introduced the OBPI based on the work of Black and Scholes [7] and Merton [38]. The investor has initial wealth W_0. He could use part of his wealth to purchase a reference portfolio (market portfolio) and the remainder to buy its put option. As the price or premium of the put option depends on market conditions, the allocation between the reference portfolio and its put option could be determined by

$$\begin{aligned} W_0 &= \omega S_0 + \omega V_{p_0} \\ &= \omega S_0 + \omega \max(K/\omega - S_T, 0), \end{aligned}$$

where ω is the allocation to the reference portfolio; V_{p_0} is the premium of the put option with the strike price of K/ω at time 0; W_0 is the value of the wealth and S_0 is the price of the reference portfolio at time 0. K, in this case, is the floor for the investor who wants to protect the minimum value of the wealth when the terminal value of the reference portfolio S_T is lower than the initial value. Then, $\max(K/\omega - S_T, 0)$ indicates the terminal payoff of the put option. The terminal wealth W_T is then expressed as

$$W_T = \begin{cases} \omega S_T & \text{if } S_T \geq K, \\ \omega S_T + K - \omega S_T = K & \text{if } S_T < K. \end{cases}$$

When the value of the portfolio is lower than the strike price, the investor exercises the option so that the loss of the portfolio is covered by the proceeds from the put option. Otherwise, the investor lets the option expire.

An investor can synthesize the put option by dynamically allocating a short position in a reference portfolio and a long position in a risk-free asset. The investor holds $\Delta(S_t, t)$ shares of the reference portfolio to replicate the put option when the price of the reference portfolio at time t is S_t. $\Delta(S_t, t)$ is the first derivative of the value of a put option at time t with respect to the value of the reference portfolio. It is the measure of the change in the option value with respect to instantaneous changes in the value of a reference portfolio.

In [37], Merton formulated his problem as

$$\text{Maximize} \left[E \int_0^T U(C, t) \mathrm{d}t \right],$$

where U is the utility function, C is consumption, E is the expectation, and T is the terminal date. Suppose markets grow as $T \to \infty$ under a stationary price movement. A long-term investor who has a constant relative risk aversion (CRRA)[4] utility function, a special type of the HARA[5] utility functions, invests in a risky asset with a constant proportion of his wealth.

The CPPI introduced by Perold [39], Black and Jones [5], and Black and Perold [6], in which the allocation of a reference portfolio e_t over an investor's wealth W_t would be determined by the wealth minus the floor level K. This difference is known as a cushion C_t and a constant multiplier m,

$$e_t = m \frac{C_t}{W_t} = m \frac{W_t - K}{W_t}. \tag{1}$$

The payoff function has concavity. When the value of the cushion decreases, the allocation to the risky assets will decrease to protect the value of the portfolio. When the multiplier is one and the floor level is equal to the value of the risk-free asset, it turns out to be the buy-and-hold strategy. As the multiplier increases, the payoff delivered by the CPPI will close as in a stop-loss strategy. Since the CPPI is equivalent to a perpetual American call with dividends, it maximizes an investor's utility under intertemporal consumption.

In [6], it is noted that transaction costs and borrowing limits complicate analysis of the CPPI, however, in practice decisions must include these two factors for the analysis to capture reality.

A large number of insurance strategies available to investors seeking downside protection are divided into several categories:

Time-dependent option replication strategies

A time-dependent option may be defined as an option whose terminal payoff is determined by time and the terminal price of an underlying asset. Simple options such as puts and calls belong to this category.

[4]

$$U(C) = \begin{cases} \frac{C^{1-\gamma} - 1}{1-\gamma} & \gamma > 0, \gamma \neq 1 \\ \log(C) & \gamma = 1, \end{cases}$$

where γ is non-negative constant.

[5]

$$U(C) = \frac{1-\gamma}{\gamma} \lambda^\gamma \text{ , } \alpha > 0 \text{ and } \lambda = \frac{\alpha C}{1-\gamma} + \beta > 0$$

Time-invariant portfolio insurance

Brennan and Schwartz [10] analyzed the class of portfolio insurance strategies independent of time. Note that most institutional investors have a portfolio without any terminal date. Such investors may need a smooth, concave payoff pattern rather than one kinked at maturity, like puts and calls. The CPPI is a special class of the time-invariant portfolio insurance.

General insurance policies

Brennan and Solanki [11] and Leland [31] suggested a general insurance policy in which the insured portfolio's payoff curve has a convex function of reference portfolio terminal value. Investors do not need to hold a fully hedged portfolio but have increasing protection as the price of the holding portfolio decreases.

Portfolio insurance strategies with specific features

- Rolling over portfolio insurance: in practice, a portfolio insurance strategy may be reset annually for easier performance review.[6]
- Dollar cost averaging portfolio insurance: periodically, the amount of investment under a portfolio insurance strategy increases by the same dollar value.[7]
- Controlling drawdown: some portfolio insurance has the additional protection from a drawdown,[8] that is, the market decline from the maximum value. In this strategy, the expected minimum value of a portfolio or strike price is adjusted over time. The level of protection is determined by an investor's utility or algorithm of financial economics.
- Non-negativity constraints: American style put options are used to protect the floor level at any time before maturity.[9]
- Exotic options: dynamic fund protection[10] delivers an automatic adjusting mechanism to lower the protection level if the fund value falls below a certain threshold level. Several exotic options including lookback options and Asian options are analyzed for an alternative investment strategy for

[6]Dybvig [18] concluded that rolling over portfolio insurance was inefficient.

[7]Brennan and Solanki [11] analyzed the optimality of investment made by dollar cost averaging portfolio insurance and concluded that such a policy was not optimal.

[8]see Grossman and Zhou [22], and Roche [43]. Grossman and Zhou defined the drawdown as

$$1 - \frac{W_t}{M_t},$$

where W_t is the wealth of a risky portfolio and M_t is the maximum value of the portfolio between time zero and time t.

[9]Karoui, Jeanblanc and Lacoste [23] discusses the optimality of American style put option portfolio insurance strategy.

[10]see [29]

simple portfolio insurance.[11] The option on a CPPI[12] is a closed-form solution for the price of a call on the CPPI without early exercise.

3 Complete and less complete markets

A complete market is a market in which cash flows generated by a trading strategy can be replicated by using different trading strategies. Arrow and Debreu [2] and McKenzie [40] contributed the proof of the existence of a complete market.

The Capital Asset Pricing Model(CAPM) is a historic break-through in financial economics. It explains the presence of the risk premium. Another milestone is the BS option pricing formula[7][38], the closed-form solution for the value of an option on stock without dividends in a risk-neutral world. These two models have common assumptions:

- The logarithmic returns of the underlying asset are normally distributed with a constant drift and volatility.
- There is no arbitrage.
- Markets are frictionless.

3.1 Complete markets

To simplify the discussion, we assume that the value of a stock follows a binomial random walk over one period[16]. The initial value of the stock is S_0. Its value at the end of the period is S_0u or S_0d depending on the state: u(up) or d(down), respectively. The returns on the stock are denoted by u and d for states 1 and 2, respectively. The current value of the call on the stock is traded at V_{c0}. At the end of the period, its value will be V_{c_1} or V_{c_2} when the value of the stock becomes S_0u or S_0d, respectively. Thus $V_{c_1} = \max(0, S_0u - K)$ and $V_{c_2} = \max(0, S_0d - K)$ where K is a strike price of the call. The return on a risk-free asset r_f is constant and the same for state 1 and state 2. The initial value of the risk-free asset, B_0, is $B = B_0r_f$ at the end of period. Thus the pricing process of the risk-free asset is deterministic. Suppose we can synthesize the call by holding ϕ_1 units of the stock and ϕ_2 units of risk-free assets

$$V_{c0} = \phi_1 S_0 + \phi_2 B_0. \tag{2}$$

At the end of the period, the value of the replicating portfolio is:

$$V_{c_1} = \phi_1 S_0 u + \phi_2 B_0 r_f, \text{ or} \tag{3}$$
$$V_{c_2} = \phi_1 S_0 d + \phi_2 B_0 r_f, \tag{4}$$

depending on the terminal state. Rearranging these formulae, we get

$$\phi_1 S_0 = \frac{V_{c_1} - V_{c_2}}{u - d} \text{ and } \phi_2 B_0 = \frac{uV_{c_1} - dV_{c_2}}{(u - d)r_f}. \tag{5}$$

[11] see Gatzent and Schmeiser [21], Leland [32]
[12] It is introduced by Escobar, et al. [20].

From (2) and (5), we find

$$V_{c0} = \left(\frac{r_f - d}{u - d} V_{c_1} + \frac{u - r_f}{u - d} V_{c_2}\right) \frac{1}{r_f}. \tag{6}$$

The above formula provides distinctive insights:

- The formula does not contain the expected value of the stock. Even though investors are heterogeneous in terms of the expected value of the stock, the hedging or replicating strategy is independent of such views.
- The option valuation formula does not have any variable related to the risk preference of the investors. The value of the call is independent of the investors' attitude toward risk.
- Due to the condition $u > r_f > d$, $(r_f - d)/(u - d)$ has a range from zero to one; it has the properties of a probability. Let it denote q, thus $q \equiv (r_f - d)/(u - d)$. Rearranging the formula, $r_f = qu + (1 - q)d$. q is known as a risk-neutral probability where the expected return on assets is equal to the return on the risk-free asset. Cox and Ross [12] introduced the concept of a risk-neutral world.
- If the value of V_{c0} is greater than $\phi_1 S_0 + \phi_2 B_0$, the strategy is to sell the call and replicate the long call to hedge the position of the short call. It is an arbitrage opportunity. Immediately after arbitragers find such opportunities, they continue their procedure until such supply is eliminated. Thus, the arbitrage opportunity disappears in the efficient market.

We can express (6) as a generalized form

$$V_{c0} = (q_1 V_{c_1} + q_2 V_{c_2})/r_f = \sum \psi_j V_{c_j} = \psi \sum q_j V_{c_j} = \frac{E^Q(V_c)}{r_f}, \tag{7}$$

where q_j is a risk-neutral probability corresponding to each state j, ψ_j is a state price for state j, and E^Q represents an expected value operator under the risk-neutral probability. q_j is written as

$$q_j = \frac{\psi_j}{\sum \psi_j} = \frac{\psi_j}{\psi} = \psi_j r_f. \tag{8}$$

From (7), we find

$$1 = \sum \psi_j r_{i,j} = \psi \sum q_j r_{i,j} = \psi E^Q(r_i),$$

where $r_{i,j}$ is the return on asset i for state j; ψ_j is also known as a discount probability. This shows that the price process is a martingale given the risk-neutral probability.

The probability corresponding to each state j is a state probability denoted by p_j

$$E(r_i) = \sum p_j r_{i,j}.$$

In this case, the risk-neutral probability and the state probability are equivalent measures. We introduce a pricing kernel π_j : $\psi_j = p_j \pi_j$. π_j transfers a state probability to the risk-neutral probability. The above formula is written as

$$1 = \sum \psi_j r_{i,j} = \sum p_j \pi_j r_{i,j} = E(\pi r_i) = E(\pi)E(r_i) + \mathrm{cov}(\pi, r_i). \tag{9}$$

We apply this formula to the risk-free asset

$$1 = \sum \psi_j r_{f,j} = \sum p_j \pi_j r_{f,j} = E(\pi r_f) = E(\pi)E(r_f). \tag{10}$$

From (8) and (9), we find

$$\frac{q_j}{p_j} \quad = \quad r_f \pi_j.$$

Now we can express the value of the call by introducing pricing kernel

$$V_{c0} \quad = \quad p\pi_1 \frac{V_{c_1}}{r_f} + (1-p)\pi_2 \frac{V_{c_2}}{r_f} = \frac{E(\pi V_c)}{r_f} = \frac{E^Q(V_c)}{r_f}. \tag{11}$$

Given the unique martingale probability measure, the market is complete.

3.2 Less complete markets

In a one-period, pure exchange economy, a representative investor consumes one product at the beginning and at the end of the period. The investor allocates his initial wealth W_0 between consumption C_0 and the investment of a portfolio at the beginning of the period. θ_i denotes his allocation to security i and $S_i(0)$ represents the value of security i at that time. Then $C_0 = W_0 - \sum \theta_i S_i(0)$. Security i pays dividend x_i at the end of period. The investor consumes all his final wealth $C = W = \sum \theta_i x_i$ at the end of the period. x_i is known as a terminal payoff. Investor utility is represented by the quadratic form $U(C) = 0.5C^2$.

The utility of the representative investor is given by

$$U(C_0, C) \quad = \quad U(C_0) + \rho E[U(C)] = U[W_0 - \sum \theta_i S_i(0)] + \rho E[U(\sum \theta_i x_i)],$$

where ρ is a impatience parameter. Differentiating the utility function with respect to θ_i to maximize the utility, the first order condition is written as

$$\frac{\partial U(C_0, C)}{\partial \theta_i} = -S_i(0) \frac{\partial U(W_0 - \sum \theta_i S_i(0))}{\partial C_0} + \rho E\left[\frac{\partial U(C)}{\partial C} x_i\right] = 0.$$

The value of security i at the beginning of period is proportional to the expected value of marginal utility and the terminal payoff

$$S_i(0) \quad = \quad \rho \frac{E\left[U'(C) x_i\right]}{U'(C_0)}, \tag{12}$$

where $U^{'}(C) = [\partial U(C)]/\partial W$ and $U^{'}(C_0) = [\partial U(C_0)]/\partial C_0$. Rearranging the above formula, we find the expected return on security i

$$E(r_i) = \frac{U^{'}(C_0)}{\rho E[U'(C)]} - \frac{\text{cov}[U^{'}(C), r_i]}{E[U'(C)]}, \tag{13}$$

where $r_i = x_i/S_i(0)$. When a risk-free asset is traded in markets, the above equation is transformed into the form of the return on the risk-free asset

$$r_f \quad = \quad \frac{U^{'}(C_0)}{\rho E[U'(C)]}.$$

Substituting r_f in (13), we get

$$E(r_i) - r_f \quad = \quad -\text{cov}[U^{'}(C), r_i]/E[U^{'}(C)]. \tag{14}$$

From the assumption of $U(C) = 0.5C^2$, $U^{'}(C) = C = \sum \theta_i S_i$. The allocation of the portfolio is optimal when the correlation of C and the aggregate wealth is equal to one. This portfolio is known as a market portfolio, denoted by r_M. We can rewrite (14) to

$$E(r_M) - r_f \quad = \quad -\text{var}(r_M)/r_M. \tag{15}$$

From (14) and (15), we have

$$E(r_i) - r_f \quad = \quad b[E(r_M) - r_f],$$

where $b = \text{cov}[r_M, r_i]/\text{var}(r_i)$. This is the formula of the CAPM. It explains that the risk premium per unit of risk of asset i adjusted by the correlation between the asset and the market portfolio must be equal to the risk premium per unit of risk of the market portfolio.

If we replace

$$\rho \frac{U^{'}(C)}{U'(C_0)} = \pi_i \tag{16}$$

in (12), we can find

$$S_i(0) \quad = \quad E(\pi_i x_i) = E^Q(x_i).$$

Thus, the above formula implies that the pricing kernel is equal to the marginal rate of substitution between future consumption and current consumption.

The major properties of the complete market are the presence of risk-neutral investors and the equipartition of states . In less complete markets, the demand and supply of a security are at equilibrium with a wide variety of risk averse investors.

4 Properties of a portfolio insurance strategy

The properties of portfolio insurance might be applicable to the most dynamic investment strategies. If a dynamic investment strategy satisfies the conditions of "self-financing", it might be path-independent and optimize the investor's utility function. The path-independency has a strong link with the delta, gamma and theta of options.

4.1 Descriptive statistics

In the binomial option model discussed in sub-section 3.1, the expected return on the stock and its variance are given by

$$\begin{aligned} E(r_s) &= p \cdot u + (1-p) \cdot d, \\ \mathrm{var}(r_s) &= p(1-p)(u-d)^2, \end{aligned}$$

where r_s is the return on the stock. We get the expected return on portfolio insurance and its variance

$$\begin{aligned} E(r_{PI}) &= \frac{pV_{c_1} + (1-p)V_{c_2}}{V_{c0}} \qquad (17) \\ &= \phi_1 E(r_s) + \phi_2 r_f, \\ \mathrm{var}(r_{PI}) &= p(1-p)(V_{c_1} - V_{c_2})^2 \\ &= p(1-p)\left[\phi_1(u-d)\right]^2, \end{aligned}$$

where r_{PI} is the return on portfolio insurance. From (6), (11), and (17), we obtain

$$\begin{aligned} E(r_{PI}) &= \frac{pV_{c_1} + (1-p)V_{c_2}}{qV_{c_1} + (1-q)V_{c_2}} r_f \\ &= \frac{E(V_c)}{E^Q(V_c)} r_f = \frac{E(V_c)}{E(\pi V_c)} r_f. \end{aligned}$$

4.2 Self-financing

The value of wealth will fluctuate over time. Price-movement uncertainty is described by two states: up movement and down movement. From (2),(3), and (4), we get

$$\begin{aligned} V_{c0} r_f &= \phi_1 S_0[qu + (1-q)d] + \phi_2 B_0[qr_f + (1-q)r_f] \\ &= \phi_1 E^Q(S) + \phi_2 B, \qquad (18) \\ \phi E^Q(S) &= \phi S_0[qu + (1-q)d], \qquad (19) \\ \phi_2 B &= \phi_2 B_0[qr_f + (1-q)r_f]. \qquad (20) \end{aligned}$$

If the above formulae hold and the values at the end of the period are exactly same as the values for the following period, then the strategy is "self-financing".

The risky portfolio, the risk-free asset, and total wealth each earn the risk-free rate. Wealth fluctuates as the value of each portfolio moves up and down, however, if these formulae are satisfied, these value processes are martingales under the presence of the risk-neutral probability measure. Generalizing the formula, we get

$$\begin{aligned}
E(V_c) &= \phi_1 S_0[pu + (1-p)d] + \phi_2 B_0[pr_f + (1-p)r_f], \\
&= \phi_1 E(S) + \phi_2 B, \\
\phi_1 E(S) &= \phi_1 S_0[pu + (1-p)d], \\
\phi_2 B &= \phi_2 B_0[pr_f + (1-p)r_f].
\end{aligned}$$

If this set of formulae hold and the set of values at the end of the period are exactly the same as the set of values for the following period, then the strategy is self-financing, and the value processes of the reference portfolio, the risk-free portfolio, and total wealth must satisfy an Itō process. In the case of the presence of the risk-neutral martingale measure, they satisfy the form of the Black-Scholes partial differential equation (PDE)(see 4.6.1). We can say that self-financing is the law of value conservation(see [15]).

4.3 Path-independent

Look at Equations (18), (19), and (20) carefully. If the value of the position of a reference portfolio first goes up and then down or first goes down then up, in both cases the position of the reference portfolio reaches the same value. We can observe the same mechanism for the value of wealth. Thus, if a dynamic investment strategy satisfies the condition of self-financing, then it is also path-independent. The terminal values depend only on the final state and are independent of the process to reach it .

4.4 Optimality of investor's expected utilities

We now consider the n-step period that contains j steps of upward moves and $n-j$ steps of downward moves, denoted by (n, j). The probability of state (n, j) is

$$p(n, j) = \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j}.$$

The corresponding state price is

$$\psi(n, j) = \frac{n!}{(n-j)!j!} \frac{q^j (1-q)^{n-j}}{r_f^n}.$$

If the strategy is a self-financing, then $p(n, j)$ and $\psi(n, j)$ are path-independent. From these two equations, we get

$$\pi(n, j) = \frac{\psi(n, j)}{p(n, j)} = \left[\frac{q(1-p)}{p(1-q)}\right]^j \left[\frac{1-q}{(1-p)r_f}\right]^n.$$

The pricing kernel $\pi(n, j)$ decreases as the number of upward steps increases when the real state probability is higher than the risk-neutral probability, $p > q$, because an investor with a long position will make money. The pricing kernel increases as j increases when $p < q$. There is no investment where $p = q$. Thus, the presence of $\rho > 0$[13] and from (12) and (16) implies that the strategy will achieve a strictly increasing utility function when $p > q$(see [15] and [17]). The strategy will optimize a risk-averse investor's expected utility.

4.5 Efficiency

Dybvig [18][19] introduced an efficiency measure for a dynamic investment strategy, the payoff distribution pricing model(PDPM).

Now, going back to subsection 3.1, the value of the call is

$$E^Q(V_c) = qV_{c_1} + (1-q)V_{c_2}.$$

Now we replace the payoff of V_{c_1} and V_{c_2} and compute the distribution price V_c^*

$$E^Q(V_c^*) = qV_{c_2} + (1-q)V_{c_1}.$$

The efficiency loss of the payoff V_c is defined as

$$E^Q(V_c) - E^Q(V_c^*).$$

If $E^Q(V_c) - E^Q(V_c^*) < 0$, then V_c is efficient, otherwise, we confirm the presence of another strategy that has the same distribution as V_c under the state probability measure but is cheaper. If $E^Q(V_c) - E^Q(V_c^*) > 0$, we would be better off having a put with strike price K.

Dybvig concluded that rolling over portfolio insurance was not efficient. Bernard et al. [8] and Vanduffel et al.[46] analyzed several dynamic investment strategies by the PDPM and found that the CPPI with discrete rebalancing was not efficient.

4.6 Time-dependency

A dynamic investment strategy that allocates the fund between a reference portfolio and a risk-free asset to achieve a given payoff $x(S_T, T)$ at terminal date T is classified as a time-dependent strategy. Its value V is a function of time and the value of the reference portfolio:

$$V(S_t, t) = E^Q[x(S_T, t)].$$

The price of the reference portfolio follows a geometric Brownian motion with stationary mean μ and variance σ^2,

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}z_t, \tag{21}$$

[13]see section 3.2.

where z is the increment to a standard Brownian motion. V satisfies the Black-Scholes partial differential equation (PDE)(see 4.6.1) and the allocation to the reference portfolio at time t.

Time invariant portfolio insurance is defined as a type of portfolio insurance investment strategy whose exposure to the reference portfolio is a function of its value. Brennan and Schwartz [10] analyzed the whole class of portfolio insurance strategies Δ that were independent of time

$$\lim_{T\to\infty} \frac{E^Q[x'(S_T)S_T]}{E^Q[x(S_T)]} = \Delta(S_t).$$

In a time invariant strategy, we can set the constant exposure above or below a predetermined price level

$$\begin{aligned}
\Delta(S_t) &= b \text{ if } S_t > S^*, \\
&= m\frac{C_t}{W_t} \text{ if } S^{**} \leq S_t \leq S^* \text{ in case of the CPPI} \\
&= 0 \text{ if } S_t < S^{**},
\end{aligned}$$

where S^* is an upper bound, S^{**} is a lower bound, and b is a constant and represents the maximum exposure.

4.6.1 Option-based portfolio insurance (OBPI)

Leland and Rubinstein [34] introduced the technique to replicate the pay-off of an option by using the Black-Scholes option pricing model. For any time $0 \leq t \leq T$, the value of a put V_p is written as

$$\begin{aligned}
V_p(S_t, t) &= E^Q[\max(K - S_T, 0)] \\
&= -\left[S_t N(-h) - Ke^{-r(T-t)}N(\sigma\sqrt{T-t} - h)\right],
\end{aligned}$$

where $N()$ is the standard cumulative normal distribution and $h = [\log(S_t/K) - r(T-t)]/\sigma\sqrt{T-t} + 1/2\sigma\sqrt{T-t}$. The put can be replicated by holding:

$$\begin{aligned}
&\Delta_p(S_t, t)S_t && \text{of the reference portfolio,} \\
&V_p(S_t, t) - \Delta_p(S_t, t)S_t && \text{of the risk free asset maturing at } T,
\end{aligned}$$

where

$$\Delta_p(S_t, t) = \frac{\partial V_p(S_t, t)}{\partial S}.$$

Thus, an insured portfolio will contain:

$$\begin{aligned}
&\Delta_p(S_t, t)S_t + S_t && \text{of the reference portfolio, and} \\
&V_p(S_t, t) - \Delta_p(S_t, t)S_t && \text{of the risk-free asset maturing at } T.
\end{aligned}$$

That the investor should hold the reference portfolio over his wealth at time t is given by

$$\omega_t = \frac{\Delta_p(S_t, t)S_t + S_t}{V_p(S_t, t) + S_t}.$$

The value of the OBPI is a function of time and the value of the underlying assets; $V(S_t, t), t \leq T$. The exposure of the reference portfolio must be rebalanced over time even though the value of the portfolio is stable; this is a property of the time-dependent portfolio insurance strategy.

4.6.2 Constant proportion portfolio insurance(CPPI)

Initial surplus or the cushion C_0 is the initial wealth W_0 less floor K. The exposure of the reference portfolio is given by (1). When $e_t > W_t$, the investor borrows funds in amount of $e_t - W_t$. Otherwise, the amount $B_t = W_t - e_t$ is invested into risk-free assets: $\mathrm{d}B_t = B_t r_f \mathrm{d}t$ due to a constant risk-free rate.

The value of cushion will be written as

$$\mathrm{d}C_t = \mathrm{a}(C_t, t)\mathrm{d}t + \mathrm{b}(C_t, t)\mathrm{d}z.$$

It must be

$$\begin{aligned} \mathrm{d}C_t &= \mathrm{d}(W_t - K) \\ &= (1 - e_t)W_t \mathrm{d}B_t/B_t + e_t W_t \mathrm{d}S_t/S_t - \mathrm{d}K \\ &= C_t \left[(1 - m)\mathrm{d}B_t/B_t + m\mathrm{d}S_t/S_t\right]. \end{aligned} \tag{22}$$

The value of the reference portfolio follows a random walk. From (21) and (22), and assuming frictionless and continuous trading, $a(C_t, t) = C_t \left[m(\mu - r_f) + r_f\right] t$, and $b(C_t, t) = C_t m\sigma$. Then applying Itō's lemma with $\ln C$[14] :

$$C_t = C_0 \exp\left\{\left[m(\mu - r_f) + r_f - \frac{m^2\sigma^2}{2}\right] t + m\sigma z_t\right\}.$$

Now applying Itō process with $\ln S$[15], we obtain

$$z_t = \frac{1}{\sigma}\left[\ln\left(\frac{S_t}{S_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right) t\right].$$

[14] $H = \ln C_t$, then we find

$$\begin{aligned} dH &= \left(\frac{\partial H}{\partial C} aC + \frac{\partial H}{\partial t} + \frac{\sigma_t^2}{2}\frac{\partial^2 H}{\partial C^2}\right)\mathrm{d}t + \frac{\partial H}{\partial C} bC\mathrm{d}z_t \\ &= (a - b^2/2)\mathrm{d}t + b\mathrm{d}z. \end{aligned}$$

[15] $G = \ln S$, thus

$$\begin{aligned} dG &= \left(\frac{\partial G}{\partial S}\mu_t S + \frac{\partial G}{\partial t} + \frac{\sigma_t^2}{2}\frac{\partial^2 G}{\partial S^2}\right) dt + \frac{\partial G}{\partial S}\sigma_t S\mathrm{d}z_t \\ &= \sigma\mathrm{d}z + (\mu - \sigma^2/2)\mathrm{d}t. \end{aligned}$$

Substituting this formula into the formula for C_t, we get

$$\begin{aligned} C(S_t) &= C_0\left(\frac{S_t}{S_0}\right)^m \exp\left\{\left[m(\mu - r_f) + r_f - \frac{m^2\sigma^2}{2}\right]t + m\left(\mu - \frac{1}{2}\sigma^2\right)t\right\} \\ &= C_0\alpha_t\left(\frac{S_t}{S_0}\right)^m, \end{aligned}$$

where

$$\alpha_t = \exp\left\{\left[r - m\left(r - \frac{1}{2}\sigma^2\right) - m^2\frac{\sigma^2}{2}\right]t\right\}.$$

The value of the CPPI payoff is

$$W(S_t) = Ke^{rt} + C_0\alpha_t\left(\frac{S_t}{S_0}\right)^m.$$

It does not depend on t but S_t.

The price sensitivity of the value function of $W(S_t)$ is given by

$$\Delta_{cppi}(S_t) = \frac{\partial W(S_t)}{\partial S} = \alpha m C_0\left(\frac{S_t}{S_0}\right)^{m-1}.$$

It is independent of time t. As long as the price of a reference portfolio is stable, the investor does not need to rebalance the positions.

The above analysis is valid when S is below the price level of maximum exposure.

4.7 Predictability of volatility charge

An investment strategy for portfolio insurance might be trade-intensive. To achieve its target payoff, a dynamic investment strategy needs to rebalance trading positions. When the price of a reference portfolio increases, the exposure will increase and when the price decreases, the exposure will decrease. Since rebalancing is the continuous operation of buying high and selling low, market fluctuations present a cost of the strategies. This is called a volatility charge.

Predictability of the volatility charge depends on the structure of the replication strategy. Even when there is extensive scenario analysis, the performance of a replication strategy cannot be controllable if the volatility charge is not predictable.

4.7.1 Option based portfolio insurance (OBPI)

Let's say that $V_c(S,t)$ is the price of a call. Then applying Itō's lemma, the function V_c of S_t and t is

$$\mathrm{d}V_c = \left(\frac{\partial V_c}{\partial S}\mu S + \frac{\partial V_c}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V_c}{\partial S^2}\right)\mathrm{d}t + \frac{\partial V_c}{\partial S}\sigma S\mathrm{d}z. \tag{23}$$

The value of a call fluctuates up-and-down randomly because of the $\mathrm{d}z$ term.

Consider the trading strategy that mitigates the risk of holding an option in any one moment. That is, only take into account the sensitivity of the option to price movements. We know Δ as the measure of the sensitivity with respect to the price changes. Holding some amount of long option V_c and $\Delta_c(S,t) = -\partial V_c/\partial S$ shares of the underlying stock, the value of this portfolio W will be

$$W = V_c - \Delta_c(S,t)S. \tag{24}$$

This strategy is known as a delta-hedge. The value of the portfolio will move from time to time depending on how the stock price moves. If the stock price moves, we will instantaneously adjust the trading position to mitigate the risk.

From time t to $t + \Delta t$, instantaneous profit and loss can be written as

$$\frac{\Delta W}{\Delta S} = \frac{\Delta V_c}{\Delta S} - \frac{\partial V_c}{\partial S}. \tag{25}$$

During the adjustment process, we will accumulate the profit and loss, that is

$$\Delta W = \Delta V_c - \frac{\partial V_c}{\partial S}\Delta S.$$

By substituting (21) for ΔS and (23) for ΔV_c, we can obtain

$$\Delta W = \left(\frac{\partial V_c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_c}{\partial S^2}\right)\Delta t.$$

This formula does not contain a $\mathrm{d}z$ term, thus it is deterministic and absolutely risk-free. This kind of portfolio is known as delta neutral. Since it is a riskless portfolio, it must earn the risk-free rate, that is,

$$r_f W = \frac{\Delta W}{\Delta t} = \left(\frac{\partial V_c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_c}{\partial S^2}\right),$$

otherwise there is an arbitrage opportunity. By substituting (24) for W, we obtain the Black-Scholes partial differential equation (PDE)

$$r_f V_c = \frac{\partial V_c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_c}{\partial S^2} + r_f S \frac{\partial V_c}{\partial S}.$$

The BS option model is the closed-form solution of this formula. As the result the option premium is equal to the expected value of the volatility cost. Thus, the volatility cost is predictable for replicating a call if this set of assumptions is met.

4.7.2 Constant proportion portfolio insurance (CPPI)

Black and Perold [6] analyzed the volatility cost of the CPPI with discrete rebalancing. A trader will rebalance the position depending on an up or down-move in the reference portfolio. The size of the up-move and down-move are denoted as u and d, respectively. Both are positive and constant. The relationship between the size of the up-move and the size of the down-move is given by $(1+u) = 1/(1-d)$. A sequence of price movements is described as the number of up-moves and down-moves, denoted by i and j, respectively,

$$S = S_0(1+u)^i(1-d)^j. \tag{26}$$

After one up-move, the change to cushion C is

$$\mathrm{d}C = nS_0(1+u) - nS_0 = nS_0u = mC_0u,$$

where n is the number of units of the reference portfolio and $mC_0 = nS_0$. Thus the cushion is given by

$$C_1 = mC_0u + C_0 = C_0(1+mu).$$

It is followed by one down-move, then

$$C_2 = C_1(1-md) = C_0(1+mu)(1-md).$$

The cushion C_0 is not equal to C_2 except when $m = 1$. If m is greater than 1, $(1+mu)(1-md) < 1$. A pair consisting of an up-move and a down-move is known as a reversal. The cost of one reversal is

$$\alpha = (1+mu)(1-md),$$

and the value of the cushion C after i up-moves and j down-moves is given by

$$C = C_0(1+mu)^i(1-md)^j. \tag{27}$$

From (26) and (27) and eliminating i and j, the value of the cushion is given by

$$C = C_0\alpha^{0.5n}(S/S_0)^\gamma,$$

where $\gamma = 0.5\ln[(1+mu)/(1-md)]/\ln(1+u)$ and $n = i+j$.

Even though the assumptions are met, the volatility cost is not predictable for the CPPI with discrete rebalancing because of varying reversal charges per price movement of the reference portfolio, except when $m = 2$. In [6] the CPPI is considered to be weak-form path-independent. As u and d are close to zero, the number of reversals will be infinite and the volatility charge turns out to be deterministic. In such cases, the CPPI is path-independent.

This analysis is only useful when the price of a reference portfolio is less than the maximum exposure.

4.8 Gamma controllability

As described in (25), the source of a replication cost occurs when the replication positions of a dynamic investment strategy are linear but the replication payoff is not linear. When the terminal payoff is not linear, the degree of its curvature is its convexity. This convexity is the second derivative of the value of a financial model with respect to an input price, like gamma for the BS model. It is the second derivative of the value of an option with respect to the underlying asset. Steep curvature of the replication payoff implies high gamma.

Consider replicating an option with a concave payoff and a hedge position that is the tangent line at a certain price. The payoff curve is always greater than the tangent line on both sides of the tangency point. In other words, the expected value of the payoff curve is greater than or equal to the function of the expected value

$$E[f(x)] \geq f[E(x)].$$

This is called the Jensen inequality. The gap between the payoff curve and the tangent line is related to the value of the option. In the BS model, ignoring interest rates and price movements, theta[16] is exactly equal to gamma. The value of the option depends on the convexity of the terminal payoff. More precisely, as the gap between the expected value of the payoff curve and the function of the expected value widens, the option price increases.

4.8.1 Time dependent options

The gamma of puts and calls is the second derivative of the value of the option with respect to the underlying asset price

$$\mathrm{Gamma(S_t, t)} = \frac{\partial \Delta(S_t, t)}{\partial S}.$$

It is a function of both the time to the maturity and price level. The gamma of puts and calls is peaked as the terminal price is closer to being the at-the-money. Changing strike price K and volatility σ can control gamma.

4.8.2 Time invariant options

The gamma of the value of the CPPI payoff is given by

$$\mathrm{Gamma}(S_t, t) = \frac{\partial \Delta_{cppi}(S_t, t)}{\partial S} = \alpha m(m-1)C_0 \left(\frac{S_t}{S_0}\right)^{m-2}.$$

It is a function of price and does not depend on the time to the maturity. Gamma becomes large as the price increases except when $m = 2$. Gamma is controlled by adjusting multiplier m and strike price K.

4.8.3 Constant Gamma

When the multiplier of the CPPI is two, then gamma becomes constant. In case of $m > 2$, the volatility cost might be high, because gamma is an increasing function of m, the moneyness, and the volatility of a reference portfolio. We can find another example of a gamma constant strategy in square power options[45].

[16] Theta is the option time value.

4.9 In state switching markets

In stationary market settings, the theory of a dynamic investment strategy has been well developed and applied to many types of asset classes. Long-term investors, however, face economies that have booms and recessions. The patterns of their income and consumption are changed over time.[17] Whenever the states of the economy or market conditions change, the investment strategy is reset. However, as the investment strategies that have a reset structure such as rolling over portfolio insurance, dollar averaging, and a stop-loss are inefficient[4][11][18], we should have an investment strategy without a reset.

The states in financial markets are grouped into a hierarchical structure [24][25][1] that has a number of clusters classified by the size of the volatility, the pattern of price movements, the degree of government intervention, and so on. The tree structure might be useful for such classifications. If the states in markets switch from one node to another, the dynamic investment strategy should also switch from one strategy to another.

In [6], the high gamma in the CPPI may reduce the number of rebalancing between the maximum exposure and zero exposure or vice versa. When the replication cost in state switching markets is considered, we have to take into account:

- As the gamma increases, the number of rebalancing decreases and the replication error per rebalancing increases.
- If the tree has a lot of branches, then the amount of information contained in each branch might decrease, that is, the profit obtained by switching strategies would be limited.
- The identification of switching between states in the market is not cost free.
- The cost of switching between investment strategies is not free.

5 Who should buy portfolio insurance?

The BS model does not tell us what type of investors should benefit from purchasing put options or insuring against downside risk. Benninga and Blume [3] state that the investor in the complete market did not have any demand for portfolio insurance or options markets. In less complete markets, an investor may need an insured portfolio.

Optimal portfolio insurance strategy was analyzed by Leland [31], and Brennan and Solanki [11]. Assuming the hypothetical representative investor supporting market prices, Leland analyzed the behavior of an individual investor who optimized his own expected utility. He concluded that two types of investors need general insurance policies:

[17] please see the section of 'Practical implications' in [4].

- An investor who has average expectations but an above-average increase in risk tolerance as wealth increases; this investor may include pension and endowment funds that focus on safe investments.

- An investor who has above-average expectations of returns but average risk tolerance; this investor can beat the markets and have positive alpha. He needs to control risk within acceptable levels for his investors.

Investors following a passive investment strategy such as pension and endowment funds might not be interested in identifying the deterministic trend of a reference portfolio. They may believe that the value of a reference portfolio follows a random walk. Upward and downward trends are considered to be stochastic. Therefore, they will choose a portfolio insurance strategy to capture reasonable gains from upside potential and to protect from downside risk.

Active managers who can deliver positive alpha through portfolio selection, market timing, and arbitrage believe that the return on their active investment is stable and predictable to a degree. Although the fluctuations of their returns would not be acceptable for some of their investors, they will mitigate such volatility by implementing portfolio insurance.

Leland and Rubinstein [34] mention that some investors became more risk averse when the value of the reference portfolio reached certain levels. Such investors will decrease their investment and withdraw funds from risky assets, shifting to cash as their investment value increases.

If states of the economy are shifting from one cluster to another in the long-term, the analysis of the demand for portfolio insurance becomes more complicated and there are a wide variety of preference patterns for the large number of different types of investors in financial markets. A preference-free approach and switching between sub-strategies may be a potential solution for realized problems.

6 Conclusions

Portfolio insurance is one of the successful applications of economic financial theory. The optimization of an investor's expected utility function, risk-neutral measure, martingale, path-independency, and payoff pricing distribution models are all essential gradients in the development of portfolio insurance. In the case of a stationary economic environment, these theories are well developed and their application might be successful, however, in an economy switching from one state to another, we are just at the starting point for theoretical development. A hierarchical structure of market states, the large deviation theory for identifying the limit of the market state, and the Kullback-Leiber divergence for a distance measure might be essential gradients in portfolio insurance in the future.

All these discussions might be applicable in any investment horizon including a long-term, short-term, or even high frequency trading, but further research is needed.

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